Continuous Variable Entropic Uncertainty Relations in the Presence of Quantum Memory

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We generalize entropic uncertainty relations in the presence of quantum memory [Nature Physics 6 (659), 2010], and [Physical Review Letters 106 (110506), 2011] in two directions. First, we consider measurements with a continuum of outcomes, and, second, we allow for infinite-dimensional quantum memory. To achieve this, we introduce conditional differential entropies for classical-quantum states on von Neumann algebras, and show approximation properties for these entropies. As an example, we evaluate the uncertainty relations for position-momentum measurements, which has applications in continuous variable quantum cryptography and quantum information theory.

Introduction. The uncertainty principle expresses the fundamental quantum feature that measurements of two non-commuting observables necessarily lead to statistical ignorance of at least one of the outcomes [16]. Entropic uncertainty relations establish a quantitative formulation of this principle. They were first studied for position and momentum operators by Hirschman [17], and subsequently improved by Beckner [1] and Bialynicki-Birula and Mycielski [5]. Deutsch stated in [10] an entropic uncertainty relation for finite-dimensional observables, which was tightened by Maassen and Uffink [20] following a conjecture of Kraus [18],

$$H(X) + H(Y) \geq \log \frac{1}{c} \ , \tag{1}$$

where $H(X)$ and $H(Y)$ are the Shannon entropies of the outcome distributions of non-degenerate measurements X and Y and $c = \max_{i,j} |\langle x_i | y_j \rangle|^2$ with $|x_i\rangle$ and $|y_j\rangle$ the eigenvectors of X and Y, respectively. The inequality was further generalized to observables described by positive operator valued measures, and to different entropies[1] (see the recent review articles [6, 31] for references).

But so far, the connection of uncertainty relations to another fundamental quantum feature, entanglement, was not fully understood. The discussion already started in the famous EPR paper in 1935 [11], but a quantitative and operationally useful criteria was missing. It was only recently realized that uncertainty should not be treated as absolute, but with respect to the knowledge of an observer [2, 15, 22]. And if the side information of the observer is quantum, one obtains a subtle interplay between the observed uncertainty, and the entanglement between the measured system and the quantum side information. Quantitatively, this can be expressed by uncertainty relations in terms of conditional entropies [2, 7–9, 22, 30]. For a bipartite state ρ_{AB} and measurements as above, we have [2]

$$H(X|B) + H(Y|B) \geq \log \frac{1}{c} + H(A|B) \ . \tag{2}$$

Here, $H(X|B)$ and $H(Y|B)$ is the conditional von Neumann entropy of the measurement outcomes of X and Y given the side information B,[2] and $H(A|B)$ is the conditional von Neumann entropy of the state ρ_{AB}. The latter quantity can be negative in the presence of entanglement, and measures the initial correlations between A and B. Because of the monogamy property of entanglement, the tripartite scenario allows a particularly elegant formulation [2, 22]: it holds for any tripartite quantum state ρ_{ABC} and measurements as above,

$$H(X|B) + H(Y|C) \geq \log \frac{1}{c} \ . \tag{3}$$

It is remarkable that the constant c is the same as in the relation without side information in (1), and most importantly, independent of the state. Beside its fundamental interest, entropic uncertainty relations with quantum side information

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[1] In [20], the inequality was already shown for general α-Rényi entropies [23].

[2] More precisely, $H(X|B)$ is the von Neumann entropy of the post-measurement state $\rho_{XB} = \sum_i \left(|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B\right) \rho_{AB} \left(|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B\right)$.

found various applications in quantum information theory, as for instance in quantum key distribution [2, 29, 30] or entanglement witnessing [2]. For that purpose, the inequality in (3) was generalized to other entropies [8], including the conditional min- and max-entropy [30]. The smooth min- and max-entropies are important one-shot entropy measures in quantum cryptography. For instance, the smooth min-entropy gives a tight characterization of privacy amplification against quantum adversaries.

However, all of the previously mentioned results involving quantum side information assume quantum systems with finitely many degrees of freedom. In contrast to this, the first papers about the uncertainty principle discuss position and momentum measurements, and thus, are for infinite-dimensional quantum systems. This problem was recently addressed by Frank and Lieb in [12]. They discuss entropic uncertainty relations with quantum side information in terms of the von Neumann entropy, which also apply to continuous position-momentum distributions. Yet, they only consider finite-dimensional quantum side information. The difficulties for defining conditional entropies in an infinite-dimensional setup were already noted by Kuznetsova [19].

Results. In this work, we derive entropic uncertainty relations with quantum side information for infinite-dimensional quantum systems without restrictions on the observables or the quantum side information. In a previous work [3], we have shown such a relation for observables with a finite outcome range for conditional min- and max-entropy. Under certain conditions this can also be extended to the von Neumann entropy by means of the asymptotic quantum equipartition property [13, 28]. Here, we follow a different approach and introduce conditional differential von Neumann $h(X|B)$ and conditional differential min- and max-entropy for classical systems described by a measure space, and the quantum side information modeled by an observable algebra. Intuitively, continuous classical systems may be thought of as being approximated by discrete systems in the limit of infinite precision. Hence, we expect a similar behavior of meaningful entropic quantities. We make this statement precise by proving that the conditional differential von Neumann as well as min- and max-entropies can be approximated by their discretized counterparts. For the von Neumann this goes as follows: let X be a classical system with range being the real line, and X_δ its restriction to a covering of $\mathbb{R}$ by half open intervals of length δ, then

$$h(X|B) = \lim_{\delta\to 0}\big(H(X_\delta|B) + \log\delta\big) \tag{4}$$

if the differential entropy $h(X|B)$ is finite. A similar result is also derived for differential conditional min- and max-entropies.

The tripartite uncertainty relation in (3) for measurements with a continuous outcome range are then derived by means of these approximation results from the finite case for the von Neumann as well as the min- and max-entropy. In particular, we prove the following main result.

Theorem 1. Let $\mathcal{M}_{ABC}$ be a tripartite von Neumann algebra, ω_{ABC} a state on $\mathcal{M}_{ABC}$, and E_X and F_Y two positive operator-valued measures acting on A. If the post-measurement states $\omega_{XBC} \equiv \omega_{ABC}\circ E_X$ and $\omega_{YBC} \equiv \omega_{ABC}\circ F_Y$ are such that $h(X|B)_\omega$ and $h(Y|C)_\omega$ are finite, then it holds that

$$h(X|B)_\omega + h(Y|C)_\omega \geq -\log c(E_X, F_Y)\ , \tag{5}$$

where the overlap of the observables is given by

$$c(E_X, F_Y) = \limsup_{\delta\to 0}\ \sup_{k,\ell\in\mathbb{Z}}\ \frac{1}{\delta^2}\left\|\sqrt{E_k^\delta}\sqrt{F_\ell^\delta}\right\|^2\ , \tag{6}$$

where $E_k^\delta = \int_{k\delta}^{(k+1)\delta} E_X(\mathrm{d}x)$ and $F_\ell = \int_{\ell\delta}^{(\ell+1)\delta} F_Y(\mathrm{d}y)$ and $\|\cdot\|$ denotes the operator norm.

In Equation (6) the indices k and ℓ can be understood as offsets whereas δ is the interval size. The operator E_k^δ is simply the POVM element corresponding to a measurement of x in the interval $[k\delta,(k+1)\delta]$, and for any interval size δ, the POVM $\{E_k^\delta\}_k$ consists of countably infinite elements.

We then discuss in detail the application of the uncertainty relation to position and momentum operators. These are the most relevant ones for continuous variable quantum cryptography as they model homodyne detection. In this case the overlaps for fixed δ are independent of the offsets k and ℓ, and can be evaluated to [27]

$$\left\|\sqrt{E_k^\delta}\cdot\sqrt{F_\ell^\delta}\right\|^2 = \frac{1}{2\pi}\delta^2 S_0^{(1)}\left(1,\frac{\delta^2}{4}\right)^2 \quad \text{and} \quad c(E_Q, F_P) = \frac{1}{2\pi}.$$

This then results in the following uncertainty relation for position and momentum operators

$$h(Q|B)_\omega + h(P|C)_\omega \geq \log 2\pi.$$

This generalizes previous results in [4, 21, 24–26].

Applications to Quantum Cryptography. A version of this uncertainty relation for conditional min- and max-entropy was recently applied to prove security of a continuous variable quantum key distribution protocol against coherent attacks including finite-size effects [14]. There, the relevant measurement was modeled using a finite number of outcomes, which simplifies the situation. Our uncertainty relation here is more general and allows modeling continuous measurement outcomes directly. We therefore believe that our uncertainty relation will provide a strong tool in continuous variable quantum cryptography.

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