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Tree-size complexity of multiqubit states

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1. INTRODUCTION

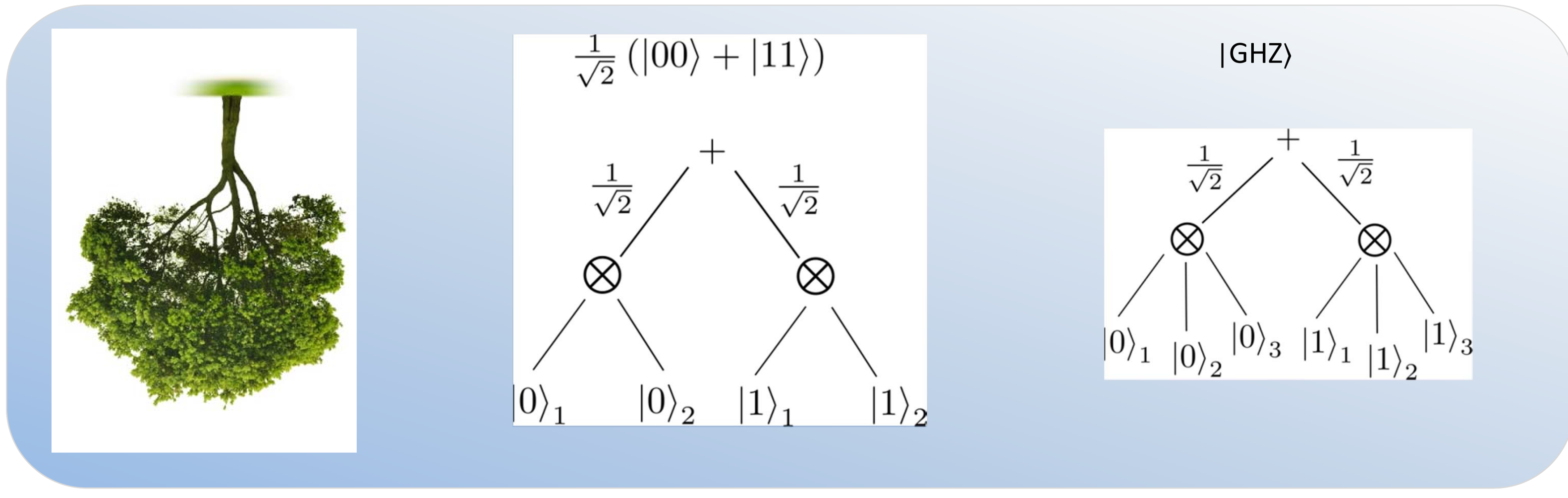
Complexity is often invoked alongside size and mass as a characteristic of macroscopic quantum objects. In 2004, Aaronson introduced the *tree size* (TS) as a computable measure of complexity and studied its basic properties. We improve and expand on those initial results. In particular, we give explicit characterizations of a family of states with superpolynomial complexity $n^{\Omega(\log n)} = \mathrm{TS} = O(\sqrt{n}!)$ in the number of qubits n.

2. MOTIVATIONS

- Testing quantum mechanics at the macroscopic scale.
- Bigger Schrodinger cats: coherent superpositions are realized with mechanical resonators, superconducting qubit, and heavy molecules.
- Complexity is an important characteristic of macroscopic systems.
- Complexity may also be relevant in the context of quantum computing: Any quantum state that offers an advantage over classical computing must be significantly complex (simple quantum states can be simulated efficiently with classical computers).
- Most states in the Hilbert space are complex, but can we write it down?
- As a starting point, we study an explicit class of superpolynomial complex quantum states (tree-size complexity is considered).

3. TREE SIZE OF A QUANTUM STATE

- Aaronson, STOC '04: Any quantum state can be described by a rooted tree of $\otimes$ and + gates. Each leaf is labeled with a single-qubit state $\alpha|0\rangle+\beta|1\rangle$.



➢ Size of a tree = number of leaves.

➢ **Tree size of a state (TS) = size of the *minimal* tree = most compact way of writing the state**

$$\underbrace{|0\rangle|0\rangle+|0\rangle|1\rangle+|1\rangle|0\rangle+|1\rangle|1\rangle}_{8\ \text{leaves}} = \underbrace{|+\rangle|+\rangle}_{2\ \text{leaves}}$$

- Tree size of some well known states:

$$TS(|0\rangle^n) = n$$
$$TS(GHZ) = 2n$$
$$TS(W) = O(n^2)$$
$$TS(1D\ cluster) = O(n^2)$$
$$TS(2D\ cluster) = 2^{\Omega(n)}\ \text{conjectured}$$
$$TS(Shor) = n^{\Omega(\log n)}\ \text{proved under one conjecture}$$

- $TS_n \leq 2^n$ for every n-qubit states (nested Schmidt decomposition).
- Upper bound on TS of a state can be obtained by finding a compact decomposition for that state $\Rightarrow$ easy to prove that some states are NOT complex.
- Any matrix-product state whose tensors are of dimension $D \times D$ has polynomial complexity $\mathrm{TS} = n^{\log_2 2D}$.
- Conjectures: If a quantum state allows universal quantum computation, it must possess superpolynomial tree size, otherwise we could simulate it efficiently.

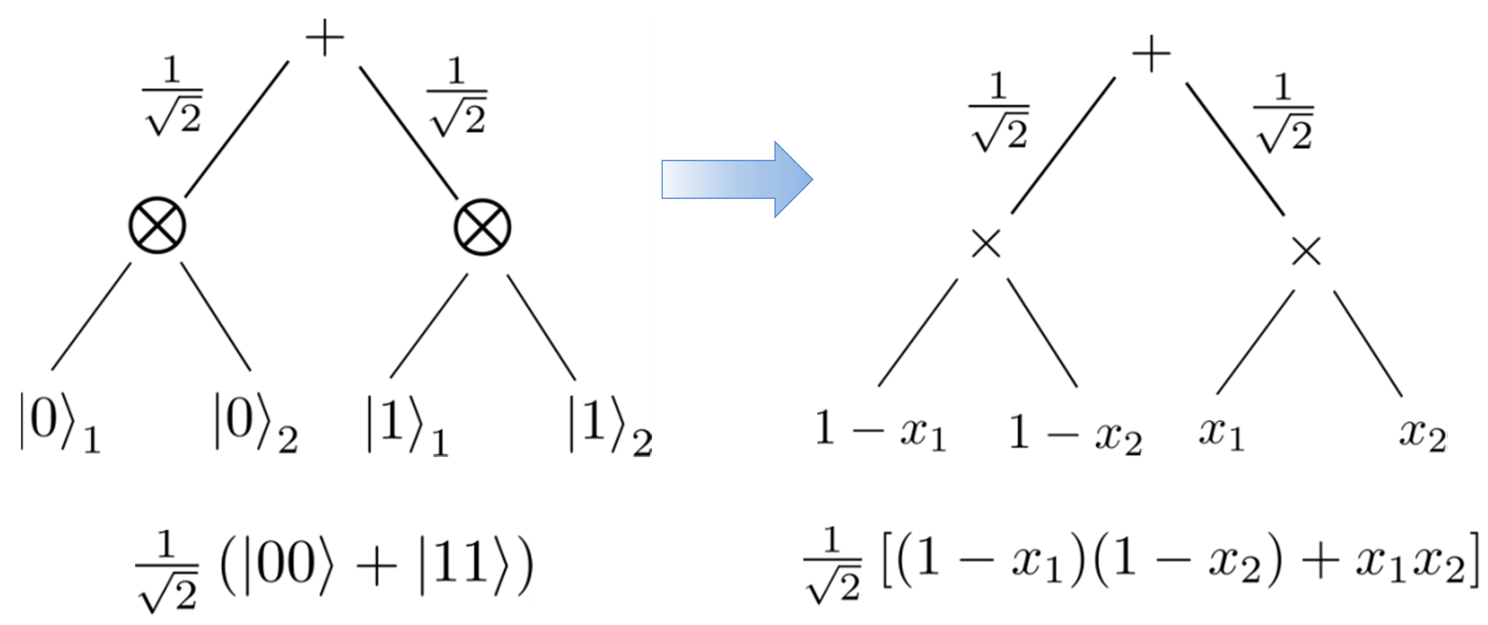
4. LINKS WITH MULTILINEAR FORMULA

- **Lower bound:** The tree size of a quantum state is bounded below by the multilinear formula size (MFS) of an associated multilinear formula.
- In an expansion of a state in the computational basis, the associated multilinear formula computes the coefficients

$$|\Psi\rangle = \sum_{x_j=0,1} f(x_1,\dots,x_n)|x_1,\dots,x_n\rangle$$

We want a multilinear formula to compute the coefficients

- To get the multilinear formula from the tree of a state: replace $|0\rangle_i$ by $1-x_i$ and $|1\rangle_i$ by x_i, $\otimes$ by $\times$



$$\mathrm{TS}(|\psi\rangle) \geq \mathrm{MFS}(f_\psi)$$

5. SUPERPOLYNOMIAL COMPLEX STATES (PRA 88, 012321)

- Raz, STOC '04: any multilinear formula that computes the **determinant** or **permanent** of a matrix is **superpolynomial**.
- When $n = m^2$, we label each computational basis vector by $|x_{11}, x_{12}, \dots, x_{mm}\rangle$, and then arrange the variables to a matrix

$$\{x\} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{pmatrix}.$$

- These states have superpolynomial tree size:

$$|\det_m\rangle = \sum_{x=0}^{2^n-1} \det(\{x\})\,|x\rangle\,,$$
$$|\mathrm{per}_m\rangle = \sum_{x=0}^{2^n-1} \mathrm{perm}(\{x\})\,|x\rangle$$

Raz | Our best decomposition

$n^{\Omega(\log n)} \leq TS \leq O((\sqrt{n})!)$

Similar to the bound "proved" for Shor states

6. MOST COMPLEX FEW-QUBIT STATES

- Tree size does not change under reversible SLOCC: All states belonging to a SLOCC-equivalent family have the same tree size.
- Tree size can be found for each SLOCC-equivalent family (practical only when the number of qubits is small).

3 qubits:

- Biseparable: TS = 5
- GHZ class: TS = 6
- W class: TS = 8

Most complex, but of zero measure:

$|W\rangle \to |W\rangle + \epsilon|111\rangle \in GHZ$ *for arbitrarily small* ϵ.

4 qubits:

- The most complex class can be written as $|0\rangle|GHZ\rangle + |1\rangle|GHZ'\rangle$ up to SLOCC; TS = 14.
- The most complex class has finite measure.
- Example: Dicke state with two excitations $|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$